Assignment 11

This homework is due *Thursday* Nov 19.

There are total 18 points in this assignment. 16 points is considered 100%. If you go over 16 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 6.1 in Bartle–Sherbert.

(1) [3pt] (Part of 6.1.1) Use the "limit of ratio" definition to find derivative of each of the following functions:
   (a) $f(x) = x^3$, $x \in \mathbb{R}$,
   (b) $f(x) = 1/\sqrt{x}$, $x > 0$.
   (c) ($\sim$6.1.2) Show that $f(x) = x^{1/10}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

(2) [3pt] (Part of 6.1.1) Find function $\varphi(x)$ involved in the Caratheodory theorem at a point $c \in \mathbb{R}$ for each of the following functions (for example, if $f = x^2$ then $x^2 - c^2 = (x + c)(x - c)$, so $\varphi(x) = x + c$):
   (a) $f(x) = x^3$, $c \in \mathbb{R}$,
   (b) $f(x) = 1/\sqrt{x}$, $c > 0$.
   In both cases, find the value of derivative at c by plugging in $x = c$ in $\varphi(x)$.

(3) [2pt] ($\sim$6.1.4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ for x rational, $f(x) = 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$. (Hint: Use the limit of ratio definition of derivative.)

(4) [3pt] (6.1.7) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c = 0$ and that $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$. (*Hint:* Use the limit of ratio definition of derivative.)

(5) [3pt] (6.1.10) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, and $g(0) = 0$. Show that g is differentiable for all $x \in \mathbb{R}$. Also show that the derivative g' is not bounded on the interval $[-1, 1]$.

(6) [2pt] (6.1.14) Given that the function $h(x) = x^3 + 2x + 1$, $x \in \mathbb{R}$, has an inverse h^{-1} on $\mathbb{R}$, find the value of $(h^{-1})'(y)$ at the points corresponding to $x = 0, 1, -1$.

(7) [2pt] (6.1.16) Given that the restriction of the tangent function tan to $I = (-\pi/2, \pi/2)$ is strictly increasing and $\tan(I) = \mathbb{R}$, let $\arctan : \mathbb{R} \to \mathbb{R}$ be the function inverse to the restriction of tan to I. Show that arctan is differentiable on $\mathbb{R}$ and $(\arctan y)' = (1 + y^2)^{-1}$ for $y \in \mathbb{R}$.